

# HEAT EQUATION FOR WEIGHTED BANACH SPACE VALUED FUNCTION SPACES

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**ABSTRACT.** We study the homogeneous equation (\*)  $u' = \Delta u$ ,  $t > 0$ ,  $u(0) = f \in wX$ , where  $wX$  is a weighted Banach space,  $w(x) = (1 + \|x\|)^k$ ,  $x \in \mathbb{R}^n$  with  $k \geq 0$ ,  $\Delta$  is the Laplacian,  $Y$  a complex Banach space and  $X$  one of the spaces  $BUC(\mathbb{R}^n, Y)$ ,  $C_0(\mathbb{R}^n, Y)$ ,  $L^p(\mathbb{R}^n, Y)$ ,  $1 \leq p < \infty$ . It is shown that the mild solutions of (\*) are still given by the classical Gauss-Poisson formula, a holomorphic  $C_0$ -semigroup.

## §1. INTRODUCTION, NOTATION AND PRELIMINARIES

In this note<sup>1</sup> Example 3.7.6 of [1, p. 154] about solutions of the heat equation via holomorphic  $C_0$ -semigroups is extended to weighted function spaces and Banach space valued functions. Our treatment is different from [1, p. 154]: instead of using Fourier transforms, direct methods are used.

Let  $w(x) := w_k(x) = (1 + \|x\|)^k$  with  $k \in \mathbb{R}_+ = [0, \infty)$ ,  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $\|x\| = (\sum_{k=1}^n x_k^2)^{1/2}$ . Then  $w \in C(\mathbb{R}^n)$  and

$$(1.1) \quad 1 \leq w(x+y) \leq w(x)w(y), \quad w(y) \leq w(x-y)w(x), \quad w(0) = 1, \\ |w(x+y)/w(x) - 1| \leq w(y)(w(y) - 1), \quad x, y \in \mathbb{R}^n.$$

Let  $Y$  be a complex Banach space and

$$(1.2) \quad wX = \{wg : g \in X\} \text{ with } X \text{ one of the spaces} \\ BUC(\mathbb{R}^n, Y), C_0(\mathbb{R}^n, Y), L^p(\mathbb{R}^n, Y), 1 \leq p < \infty.$$

Then  $wX$  is a Banach space with norm  $\|f\|_{wX} = \|f/w\|_X$  and a linear subset of  $\mathcal{S}'(\mathbb{R}^n, Y)$ ,  $wX$  is translation invariant, since  $X$  is and, with  $f = wg$ ,  $g \in X$ ,

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$f_h(x) := f(x+h)$ , one has  $f_h/w = g_h w_h/w$  with  $w_h/w \in BUC(\mathbb{R}^n, \mathbb{R})$ , using (1.1).

For any  $f : \mathbb{R}^n \rightarrow Y$ ,  $|f|(x) = \|f(x)\|$ ,  $x \in \mathbb{R}^n$ .

For  $f \in wX$  and  $\zeta \in \mathbb{C}^+ := \{z \in \mathbb{C} : \operatorname{Re} \zeta > 0\}$  define (see Lemma 1.3)

$$(1.3) \quad (G(\zeta)f)(x) := (4\pi\zeta)^{-n/2} \int_{\mathbb{R}^n} f(x-y) e^{-\|y\|^2/4\zeta} dy, \quad x \in \mathbb{R}^n.$$

Let  $\chi_\zeta(x) = (4\pi\zeta)^{-n/2} e^{-\|x\|^2/4\zeta}$ ,  $\zeta \in \mathbb{C}^+$ ,  $x \in \mathbb{R}^n$ . Then  $\chi_\zeta \in \mathcal{S}(\mathbb{R}^n)$  if  $\zeta \in \mathbb{C}^+$ ,

$$(1.4) \quad (G(\zeta)f) = \chi_\zeta * f, \quad \zeta \in \mathbb{C}^+, \quad G(0)f = f, \quad f \in wX.$$

The function  $\chi_\zeta$  is defined and  $\chi'_\zeta = \frac{d\chi_\zeta}{d\zeta}$  exists for each  $\zeta \in \mathbb{C}^+$ , thus holomorphic on  $\mathbb{C}^+$ . Moreover,  $\chi_\zeta^{(k)} = \frac{d^k \chi_\zeta}{d\zeta^k} \in \mathcal{S}(\mathbb{R}^n)$  for each  $\zeta \in \mathbb{C}^+$ ,  $k \in \mathbb{N}_0$ .

$$(1.5) \quad I = I(\zeta) = ((4\pi\zeta)^{-n/2} \int_{\mathbb{R}^n} e^{-(\|x\|^2/4\zeta)} dx = 1 \text{ for each } \zeta \in \mathbb{C}^+.$$

Indeed,  $I(\zeta)$  is holomorphic on  $\mathbb{C}^+$  with  $I = 1$  on  $(0, \infty)$ . It follows  $I = 1$  on  $\mathbb{C}^+$  by the identity theorem for complex valued holomorphic functions.

Also, for  $\zeta = re^{i\phi}$ ,  $0 \leq |\phi| < \alpha < \pi/2$ ,  $r > 0$ , for any  $x \in \mathbb{R}^n$

$$(1.6) \quad |\chi_\zeta(x)| = (4\pi r)^{-n/2} e^{-(\|x\|^2 \cos \phi)/4r} < (4\pi r)^{-n/2} e^{-(\|x\|^2 \cos \alpha)/4r}.$$

$$(1.7) \quad \text{Fourier transform } \widehat{\chi_\zeta}(x) = e^{-\zeta \|x\|^2}, \quad x \in \mathbb{R}^n, \quad \zeta \in \mathbb{C}^+.$$

Indeed, it is enough to prove the case  $n = 1$ . We have

$$\widehat{\chi_\zeta}(y) = e^{-\zeta y^2} I(\zeta, y), \text{ where}$$

$$I(\zeta, y) = (4\pi\zeta)^{-1/2} \int_{\mathbb{R}} e^{-(x+2i\zeta y)^2/4\zeta} dx, \quad y \in \mathbb{R}, \quad \zeta \in \mathbb{C}^+.$$

With  $F(x, y) := e^{-(x+2i\zeta y)^2/4\zeta}$ ,

$$\begin{aligned} \frac{\partial}{\partial y} \int_{\mathbb{R}} F(x, y) dx &= \int_{\mathbb{R}} \frac{\partial}{\partial y} F(x, y) dx = \int_{\mathbb{R}} 2i\zeta \frac{\partial}{\partial x} F(x, y) dx = \\ &= 2i\zeta \lim_{N \rightarrow \infty} (F(N, y) - F(-N, y)) = 0, \end{aligned}$$

so  $I(\zeta, y) = I(\zeta, 0) = 1$  for  $\zeta$  real  $> 0$  (e.g. [3, p. 274, Beispiel 1]), then for  $\zeta \in \mathbb{C}^+$  since  $I$  is holomorphic there.

**Lemma 1.1.** *If  $f \in wX$  respectively  $wf \in L^p(\mathbb{R}^n, \mathbb{C})$  with  $1 \leq p < \infty$ , then  $\|(f_y - f)/w\|_X \rightarrow 0$  respectively  $\|w(f_y - f)\|_{L^p} \rightarrow 0$  as  $y \rightarrow 0$ .*

*Proof.* Let  $f = wg$ , where  $g \in X$ . Then  $\|f_y - f\|_{wX} = \|w_y g_y - wg\|_{wX} = \|(w_y - w)g_y + wg_y - wg\|_{wX} \leq \|(w_y - w)g_y\|_{wX} + \|wg_y - wg\|_{wX} = \|(w_y/w - 1)g_y\|_X + \|g_y - g\|_X \rightarrow 0$  as  $y \rightarrow 0$  with (1.1) since  $\|g_y - g\|_X \rightarrow 0$  as  $y \rightarrow 0$ . The second case follows similarly.  $\square$

**Lemma 1.2.** (A) If  $f \in wL^p(\mathbb{R}^n, Y)$ ,  $wg \in L^q(\mathbb{R}^n, \mathbb{C})$  with  $1/p + 1/q = 1$  and  $1 \leq p \leq \infty$ , then  $(g * f)(x)$  exists as a Bochner integral for all  $x \in \mathbb{R}^n$ , and  $g * f \in wBUC(\mathbb{R}^n, Y)$ ; if additionally  $1 < p < \infty$  or  $f \in wC_0(\mathbb{R}^n, Y)$  and  $q = 1$ , then  $g * f \in wC_0(\mathbb{R}^n, Y)$ .

(B) If  $f \in wL^p(\mathbb{R}^n, Y)$ ,  $wg \in L^1(\mathbb{R}^n, \mathbb{C})$  with  $1 \leq p \leq \infty$ , then  $g * f(x)$  exists as Bochner integral almost everywhere in  $\mathbb{R}^n$  and  $g * f \in wL^p(\mathbb{R}^n, Y)$ .

*Proof.* (A) Since

$$(1.8) \quad \|f(y)g(x-y)\| = \|(f/w)(y)\| \|(wg)(x-y)\|(w(y)/w(x-y)) \leq \\ |f/w|(y) |wg|(x-y)w(x),$$

(1.1),  $|f/w| \in L^p(\mathbb{R}^n)$  and  $|wg|(x-\cdot) \in L^q(\mathbb{R}^n)$ , with the Hölder inequality [5, p. 34, Proposition 2] one has  $f(\cdot)g(x-\cdot) \in L^1(\mathbb{R}^n)$  for all  $x \in \mathbb{R}^n$ ,

$$(1.9) \quad \|g * f(x)\| \leq w(x) \|wg\|_{L^q} \|f/w\|_{L^p}.$$

With this

$$\|g * f(x+y) - g * f(x)\| \leq w(x) \|f/w\|_{L^p} \|w(g_y - g)\|_{L^q}, \\ \|g * f(x+y) - g * f(x)\| \leq w(x) \|(f_y - f)/w\|_{L^p} \|wg\|_{L^q}.$$

By Lemma 1.1,  $\|w(g_y - g)\|_{L^q} \rightarrow 0$  respectively  $\|(f_y - f)/w\|_{L^p} \rightarrow 0$  as  $y \rightarrow 0$  if  $1 \leq q < \infty$  respectively  $1 \leq p < \infty$ . It follows  $g * f \in wBUC(\mathbb{R}^n, Y)$  if  $1 \leq p, q \leq \infty$ . If  $p > 1, q < \infty$  or  $f \in wC_0(\mathbb{R}^n, Y)$  and  $q = 1$ , then  $(w|g|) * (|f|/w) \in C_0(\mathbb{R}^n)$  by [1, Proposition 1.3.2 b), d), p. 22]. It follows  $g * f \in wC_0(\mathbb{R}^n, Y)$ .

(B) By Young's inequality [5, p. 29],  $(w|g|) * (|f|/w) \in L^p(\mathbb{R}^n)$ . So,  $(w|g|) * (|f|/w)(x)$  is finite almost everywhere on  $\mathbb{R}^n$ . This, measurability of  $g(x-\cdot)f(\cdot)$  and (1.8) imply  $g * f(x)$  exists as a Bochner integral almost everywhere on  $\mathbb{R}^n$ . The above  $(w|g|) * (|f|/w) \in L^p(\mathbb{R}^n, \mathbb{C})$  and (1.8) give  $g * f \in wL^p(\mathbb{R}^n, Y)$ .  $\square$

**Lemma 1.3.** *Let  $f \in wX$ ,  $G(\zeta)$  defined by (1.3n) and  $g = \chi_\zeta$  or  $\chi'_\zeta := \frac{d\chi_\zeta}{d\zeta}$ ,  $\zeta \in \mathbb{C}^+$ .*

*(i)  $g * f(x)$  exist as a Bochner integral for all  $x \in \mathbb{R}^n$  and  $g * f \in wBUC(\mathbb{R}^n, Y) \cap wX$ ; if additionally  $1 < p < \infty$  or  $f \in wC_0(\mathbb{R}^n, Y)$ , then  $g * f \in wC_0(\mathbb{R}^n, Y) \cap wX$ .*

*(ii)  $G(\zeta) \in L(wX)$ .*

*(iii) If  $0 < \alpha < \pi/2$ , then*

$$(1.10) \quad \lim_{0 \neq \zeta \rightarrow 0, |\arg \zeta| < \alpha} \|\chi_\zeta * f - f\|_{wX} = 0.$$

*Proof.* (i) Since  $wg \in L^q(\mathbb{R}^n, \mathbb{C})$  for each  $1 \leq q \leq \infty$ , (i) follows by Lemma 1.2.

(ii) The operator  $G(\zeta) : wX \rightarrow wX$  defined by  $G(\zeta)f := \chi_\zeta * f$  is linear and bounded by (1.9).

(iii) With  $y = |\zeta|^{1/2}z$  and  $\theta = \frac{\zeta}{|\zeta|}$ , it follows by (1.5)

$$\chi_\zeta * f(x) - f(x) = \int_{\mathbb{R}^n} [f(x-y) - f(x)] \chi_\zeta(y) dy = \int_{\mathbb{R}^n} [f(x - |\zeta|^{1/2}z) - f(x)] \chi_\theta(z) dz.$$

Case  $X = BUC(\mathbb{R}^n, Y)$ ,  $C_0(\mathbb{R}^n, Y)$ . Let  $\varepsilon > 0$ . Since  $w\chi_\theta \in L^1(\mathbb{R}^n)$ , then using (1.1), for  $0 < |\zeta| \leq 1$ ,  $|\arg \zeta| < \alpha$  there exists  $c = c(\varepsilon, \alpha) > 0$  independent of  $\zeta$ , such that

$$I_1 = \sup_{x \in \mathbb{R}^n} \frac{1}{w(x)} \int_{\|z\| \geq c} \|f(x - |\zeta|^{1/2}z) - f(x)\| |\chi_\theta(z)| dz \leq 2\|f\|_{wX} \times \\ \int_{\|z\| \geq c} w(z) |\chi_\theta(z)| dz \leq 2\|f\|_{wX} (4\pi)^{-n/2} \int_{\|z\| > c} w(z) e^{-\|z\|^2 (\cos \alpha)/4} dz < \varepsilon.$$

Then for the above  $\zeta$

$$\|\chi_\zeta * f(x) - f(x)\|_{wX} \leq I_1 + \sup_{x \in \mathbb{R}^n} \frac{1}{w(x)} \int_{\|z\| \leq c} \|f(x - |\zeta|^{1/2}z) - f(x)\| |\chi_\theta(z)| dz \leq \\ I_1 + \sup_{x \in \mathbb{R}^n, \|z\| \leq c} \frac{\|f(x - |\zeta|^{1/2}z) - f(x)\|}{w(x)} (4\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\|z\|^2 (\cos \alpha)/4} dz = I_1 + I_2.$$

Using Lemma 1.1, there is  $\delta > 0$  such that  $I_2 \leq \varepsilon$  if  $|\zeta|^{1/2}c < \delta$ . It follows  $I_1 + I_2 \leq 2\varepsilon$  if  $0 < |\zeta|^{1/2} < \delta/c$  and  $|\arg \zeta| \leq \alpha$ .

Case  $X = L^p$ : By (i)  $\chi_\zeta * f \in wL^p(\mathbb{R}^n, Y) \cap BUC(\mathbb{R}^n, Y)$ . For  $\zeta \in \mathbb{C}^+$  with  $y = |\zeta|^{1/2}z$  using the Minkowski inequality [3, p. 251, A 92]

$$\|\chi_\zeta * f - f\|_{wL^p} = \left[ \int_{\mathbb{R}^n} \frac{\|\int_{\mathbb{R}^n} [f(x-y) - f(x)] \chi_\zeta(y) dy\|^p}{w_k^p(x)} dx \right]^{\frac{1}{p}} \leq \\ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \frac{\|f(x-y) - f(x)\|^p}{w_k^p(x)} dx \right]^{\frac{1}{p}} |\chi_\zeta(y)| dy = \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} \frac{\|f(x - |\zeta|^{1/2}z) - f(x)\|^p}{w_k^p(x)} dx \right]^{\frac{1}{p}} |\chi_\theta(z)| dz.$$

By Lemma 1.1,  $\int_{\mathbb{R}^n} \frac{\|f(x - |\zeta|^{1/2}z) - f(x)\|^p}{w_k^p(x)} dx \rightarrow 0$  as  $|\zeta| \rightarrow 0$  for each  $z \in \mathbb{R}^n$ . So, by the dominated convergence theorem as in Lemma 1.3.3 (b) of [1, p. 23] we get

the statement since

$|\chi_\theta(z)| < (4\pi)^{-n/2} e^{-||z||^2(\cos \alpha)/4} =: F(z)$  by (1.6),  $||f_{-|\zeta|^{1/2}z}||_{wX} \leq w(z)||f||_{wX}$  and  $wF \in L^1(\mathbb{R}^n)$ , if  $z \in \mathbb{R}^n$ ,  $|\arg \zeta| < \alpha$ ,  $0 < |\zeta| \leq 1$ .  $\square$

## §2. MAIN RESULTS

**Theorem 2.1.** *For  $wX$  of (1.2), the  $G$  of (1.3) is a holomorphic  $C_0$ -semigroup of angle  $\pi/2$  on  $wX$ . Its generator is the Laplacian  $\Delta_{wX} := \Delta$  on  $wX$  with domain:*

$$D(\Delta_{wX}) = \{f \in wX : \text{distribution-}\Delta f \in wX\}, \Delta = \sum_{j=1}^n \partial^2 / \partial x_j^2$$

where we identify  $wX$  with a subspace of  $\mathcal{S}'(\mathbb{R}^n, Y)$ .

*Proof.* (a): We have  $\chi_\zeta \in \mathcal{S}(\mathbb{R}^n)$  for  $\zeta \in \mathbb{C}^+$  and

$$\frac{d\chi_\zeta}{d\zeta}(x) = \Delta \chi_\zeta(x) \text{ for } \zeta \in \mathbb{C}^+, x \in \mathbb{R}^n.$$

Moreover, by Lemma 1.3  $G(\zeta)f = \chi_\zeta * f \in wX$ ,  $||\chi_\zeta * f - f||_{wX} \rightarrow 0$  as in (1.10) for all  $\zeta \in \mathbb{C}^+$ ,  $f \in wX$  and  $G(\zeta) \in L(wX)$ . Then  $\widehat{G(\zeta)f} = \widehat{\chi_\zeta} \cdot \widehat{f}$  follows as in [1, p. 154]. By (1.7),  $\widehat{\chi_{\zeta_1 + \zeta_2}} = \widehat{\chi_{\zeta_1}} \widehat{\chi_{\zeta_2}}$ . So,  $G(\zeta_1 + \zeta_2) = G(\zeta_1)G(\zeta_2)$ ,  $\zeta_1, \zeta_2 \in \mathbb{C}^+$ . This means that  $G$  is a  $C_0$ -semigroup on  $wX$ .

(b) Holomorphy of  $G : \mathbb{C}^+ \rightarrow L(wX)$ . By [1, Proposition A.3, (ii)  $\Rightarrow$  (i)], it is enough to show that for any  $f \in wX$  with  $U(\zeta) = G(\zeta)f$  the  $U$  is holomorphic on  $\mathbb{C}^+$ . Now, again by [1, Proposition A.3], holomorphy of the function  $\zeta \rightarrow w\chi_\zeta$  defined on  $\mathbb{C}^+$  with values in  $L^1(\mathbb{R}^n)$  follows, since the complex valued  $F(\zeta) = \int_{\mathbb{R}^n} w(x)\chi_\zeta(x)g(x)dx$  is continuous for each  $g \in L^\infty(\mathbb{R}^n)$  and by Morera's theorem [4, p.75], Fubini and (1.6) it is holomorphic. So to fixed  $z$  there exists  $\psi$  in  $L^1(\mathbb{R}^n)$  with  $w(\frac{\Delta \chi_\zeta}{\Delta \zeta}) \rightarrow \psi$  in  $L^1(\mathbb{R}^n)$ ; so there are  $\zeta_n \rightarrow \zeta$  with  $\frac{\chi_{\zeta_n} - \chi_\zeta}{\zeta_n - \zeta} \rightarrow \psi/w$  almost everywhere on  $\mathbb{R}^n$ ; with the holomorphy of  $\chi_\zeta(x)$  for each  $x \in \mathbb{R}^n$  one gets  $\psi/w = \chi'_\zeta$  almost everywhere and

$$(2.1) \quad ||(\frac{\Delta \chi_\zeta}{\Delta \zeta} - \chi'_\zeta)w||_{L^1} = \int_{\mathbb{R}^n} |\frac{\Delta \chi_\zeta(x)}{\Delta \zeta} - \chi'_\zeta(x)|w(x)dx \rightarrow 0 \text{ as } 0 \neq \Delta \zeta \rightarrow 0.$$

Since  $w\chi'_\zeta \in L^q(\mathbb{R}^n)$  for all  $q \geq 1$ ,  $\chi'_\zeta * f(x)$  exists with Hölder's inequality as a Bochner integral for all  $x \in \mathbb{R}^n$ . By Lemma 1.3,  $\frac{\Delta U(\zeta)}{\Delta \zeta}, \chi'_\zeta * f \in wX$ ,  $\zeta, \zeta + \Delta \zeta \in \mathbb{C}^+$ ,  $\Delta \zeta \neq 0$ . We have

$$\begin{aligned}
& \frac{\Delta U(\zeta)}{\Delta \zeta} - \chi'_\zeta * f = \left( \frac{\Delta \chi_\zeta}{\Delta \zeta} - \chi'_\zeta \right) * f \text{ and using Young's inequality [5, p. 29]} \\
& \|(\frac{\Delta \chi_\zeta}{\Delta \zeta} - \chi'_\zeta) * f\|_{wX} = \|(1/w) \int_{\mathbb{R}^n} (\frac{\Delta \chi_\zeta}{\Delta \zeta} - \chi'_\zeta)(\cdot - y) f(y) dy\|_X \leq \\
& \|\int_{\mathbb{R}^n} |(\frac{\Delta \chi_\zeta}{\Delta \zeta} - \chi'_\zeta)(\cdot - y)| w(\cdot - y) (\|f(y)\|/w(y)) dy\|_X \leq \\
& \|(\frac{\Delta \chi_\zeta}{\Delta \zeta} - \chi'_\zeta) w\|_{L^1} \|f/w\|_X.
\end{aligned}$$

With (2.1), holomorphy of  $U$  on  $\mathbb{C}^+$  follows, and

$$(2.2) \quad G'(\zeta)f = (G(\zeta)f)' = \chi'_\zeta * f, \zeta \in \mathbb{C}^+, f \in wX.$$

(c) Let  $f$ , distribution  $\Delta f \in wX$ . We have  $\frac{\partial \chi_t}{\partial t} = \Delta_x \chi_t$  on  $(0, \infty) \times \mathbb{R}^n$ ,  $\Delta_x = \sum_{j=1}^n (\frac{\partial}{\partial x_j})^2$ . So by (2.2), in  $\mathcal{S}'(\mathbb{R}^n, Y)$ ,  $t > 0$ ,

$$\begin{aligned}
(2.3) \quad \frac{dG(t)f}{dt} &= \frac{d(\chi_t * f)}{dt} = \frac{d\chi_t}{dt} * f = \\
& (\Delta \chi_t) * f = \Delta G(t)f = \chi_t * (\Delta f) = G(t)\Delta f.
\end{aligned}$$

Let  $A|D(A)$  be the generator of the  $C_0$ - semigroup  $G : \mathbb{R}_+ \rightarrow L(wX)$ , defined by Proposition 3.1.9 g) of [1, p. 115 ]; let  $\Delta$  be the Laplace operator applied to  $S \subset \mathcal{D}' := \mathcal{D}'(\mathbb{R}^n, Y)$ ; with  $wX \subset \mathcal{D}'(\mathbb{R}^n, Y)$ ,

$D := \{f \in wX : \Delta_{wX} f \in wX\}$  and  $\Delta_{wX} := \Delta|D$  are well defined. We show

$$(2.4) \quad D(A) = D, A = \Delta_{wX}.$$

$$(c.1) \quad D \subset D(A), A = \Delta_{wX} \text{ on } D:$$

If  $f \in D$ ,  $G(\cdot)f \in C([0, \infty), wX)$  by (a), with (2.3) and  $g := \Delta_{wX} f \in wX$  one has  $G(f)f - f = \int_0^t (\frac{d}{ds})(G(s)f) ds = \int_0^t G(s)g ds$ ,  $t \in \mathbb{R}_+$ . With Proposition 3.1.9 f) of [1, p. 115 ] one gets  $f \in D(A)$  and  $Af = g = \Delta_{wX} f$ .

$$(c.2) \quad D(A) \subset D :$$

With  $F(t) := (1/t) \int_0^t G(s)f ds$ ,  $t > 0$ ,  $f \in D(A)$ , one has  $F(t) \rightarrow f$  in  $wX$  as  $t \rightarrow 0$ , since  $G(t)f \rightarrow f$  in  $wX$  by (a).  $F(t) \rightarrow f$  in  $wX$  implies  $F(t) \rightarrow f$  in  $L^1_{loc}(\mathbb{R}^n, Y)$ , so  $(\Delta F(t))(\varphi) = F(t)(\Delta_x \varphi) \rightarrow f(\Delta_x \varphi) = (\Delta f)(\varphi)$  for  $\varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{C})$ .

Now by (2.5) below one has  $\Delta F(t) = (1/t)(G(t)f - f)$ ; by definition of  $D(A)$  and Proposition 3.1.9 g) [A., p. 115],  $(1/t)(G(t)f - f) \rightarrow$  some  $g$  in  $wX$ , so in  $L^1_{loc}(\mathbb{R}^n, Y)$ , so  $(1/t)(G(t)f - f)(\varphi) \rightarrow f(\varphi)$ . together one gets  $\Delta f = g$ ,  $\in wX$ ,

that is  $f \in D$ . With (c.1) this gives (2.4). It remains to show

$$(2.5) \quad \Delta \int_0^t G(s) f ds = G(t) f - f, \quad f \in wX.$$

For this, with  $f \in wX$ , with Lemma 1.3 define  $\beta(t, x) := (\chi_t * f)(x)$ ,  $(t, x) \in M := (0, \infty) \times \mathbb{R}^n$ . With Lebesgue's Dominated Convergence theorem and analogs of (1.6) for the derivatives of  $\chi_t$  one gets inductively  $\beta \in C^\infty(M, Y)$ , with

$$(2.6) \quad \partial \beta / \partial t = (\chi'_t) * f = (\Delta_x \chi_t) * f = \Delta_x \beta.$$

If  $0 < \varepsilon < t$ ,  $\Psi_\varepsilon(t, x) := \int_\varepsilon^t \beta(s, x) ds$ ,  $x \in \mathbb{R}^n$ , is well defined with  $\Psi_\varepsilon \in C((\varepsilon, \infty) \times \mathbb{R}^n, Y)$ ,  $\Psi_\varepsilon(t) := \Psi_\varepsilon(t, \cdot) \in C(\mathbb{R}^n, Y) \subset \mathcal{D}'(\mathbb{R}^n, Y)$  if  $t > \varepsilon$ . If  $\varphi \in \mathcal{D}(\mathbb{R}^n)$ , all the following integrals exist (even as Riemann integrals), with twice Fubini, partial integration and (2.6) one has

$$\begin{aligned} (\Delta \Psi_\varepsilon(t))(\varphi) &= \int_{\mathbb{R}^n} \Psi_\varepsilon(t, x) (\Delta_x \varphi)(x) dx = \int_{\mathbb{R}^n} \int_\varepsilon^t \beta(s, x) (\Delta_x \varphi)(x) ds dx = \\ &= \int_\varepsilon^t \int_{\mathbb{R}^n} \Delta_x \beta(s, x) \varphi(x) dx ds = \int_\varepsilon^t \int_{\mathbb{R}^n} (\partial / \partial s) \beta(s, x) \varphi(x) dx ds = \\ &= \int_{\mathbb{R}^n} (\int_\varepsilon^t (\partial / \partial s) \beta(s, x) ds) \varphi(x) dx = \int_{\mathbb{R}^n} (\beta(t, x) - \beta(\varepsilon, x)) \varphi(x) dx. \end{aligned}$$

This implies

$$(2.7) \quad \Delta \Psi_\varepsilon = G(t) f - G(\varepsilon) f, \in wX.$$

$G(\cdot) f : \mathbb{R}_+ \rightarrow wX$  is continuous, so  $\int_\varepsilon^t G(s) f ds \rightarrow \int_0^t G(s) f ds$  as  $\varepsilon \rightarrow 0$ . Furthermore, the Riemann sums  $\Sigma_m := \sum_1^m (G(s_j) f) (s_j - s_{j-1}) \rightarrow \int_\varepsilon^t G(s) f ds$  in  $wX$  as  $m \rightarrow \infty$ ,  $s_j = \varepsilon + j(t - \varepsilon)/m$ . Similarly  $\Sigma_m(x) := \sum_1^m \beta(s_j, x) (s_j - s_{j-1}) \rightarrow \int_\varepsilon^t \beta(s, x) ds = \Psi_\varepsilon(t, x)$  in  $Y$  as  $m \rightarrow \infty$ , for each  $x \in \mathbb{R}^n$ .

If  $K$  is compact  $\subset \mathbb{R}^n$ , then  $\sup \{ \|\Sigma_m(x)\| : m \in \mathbb{N}, x \in K \} < \infty$ , so

$$\int_{\mathbb{R}^n} \Sigma_m(x) \varphi(x) dx \rightarrow \int_{\mathbb{R}^n} \Psi_\varepsilon(t, x) \varphi(x) dx \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

As above,  $\int_\varepsilon^t G(s) f ds = \Psi_\varepsilon(t)$  follows, and then  $\Psi_\varepsilon(t) \rightarrow \int_0^t G(s) f ds$  in  $wX$ .

Therefore  $(\Delta \Psi_\varepsilon(t))(\varphi) = \int_{\mathbb{R}^n} \Psi_\varepsilon(t) \Delta_x \varphi dx \rightarrow \int_{\mathbb{R}^n} (\int_0^t G(s) f ds) \Delta_x \varphi dx =$

$(\Delta \int_0^t G(s) f ds)(\varphi)$  as  $\varepsilon \rightarrow 0$ ; since  $G(\varepsilon) f \rightarrow f$ , (2.7) implies (2.5).  $\square$

**Corollary 2.2.** *All mild solutions  $u : [0, \infty) \rightarrow wX$  of*

*(a)  $u' = \Delta_{wX} u$  are given by  $u(t) = G(t) f$  with  $f \in wX$ , they are  $C^1$ -solutions*

on  $(0, \infty)$ ,  $\in C^\infty(M, Y)$  and classical solutions of

- (b)  $\frac{\partial u(t, x_1, \dots, x_n)}{\partial t} = \sum_1^n \frac{\partial^2 u(t, x_1, \dots, x_n)}{\partial x_j^2}$  on  $M := (0, \infty) \times \mathbb{R}^n$  with  
(c)  $u(t, \cdot) \rightarrow f$  in  $wX$  as  $t \rightarrow 0$ .

Conversely, any classical solution of (b) with (c) defines a mild solution of (a) on  $[0, \infty)$ .

For the proofs of most of this see [1, Corollary 3.7.21].

**Remark 2.3.** Since the Gauss-Poisson formula (1.4) for  $G$  defines by Theorem 2.1 a holomorphic  $C_0$ -semigroup with generator  $A = \Delta_{wX}$ , with Corollary 2.2 the results of [2, Theorems 5.2/6.3, Examples 6.2] can be applied to the heat equation.

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